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Jacek Jagodziński
Wrocław School of Banking

The extended Hurwicz criterion for decision making in logistic applications

Abstract. The article addresses a new decision making criterion based on the popular Hurwicz criterion widely applied in the theory of games against nature. The author has described properties of the decision making function under this criterion, subsequently applied in a new decision making algorithm. The said approach may also be adopted as a new tactic for choosing the coefficient of optimism or verifying the correctness of one already chosen.

Key words: logistics, decision making problems, games against nature, the Hurwicz criterion, coefficient of optimism

1. Introduction

The field of knowledge referred to as decision analysis entails a systematic and organised approach enabling managers and analysts representing various areas of expertise to make decisions under conditions of uncertainty and risk. Decision making under conditions of uncertainty means that one does not know the probability of future states, however, their emergence is of random nature. For the sake of this study, the assumptions formulated by Bayes have been informally adopted, stating that if there is no difference in the probability of different states of nature, then it is envisaged that each mutually exclusive situation occurs with the same probability[1].

[1] Z. Redziak, "Wybrane aspekty podejmowania decyzji w warunkach ryzyka i niepewności", *Studia i Materiały. Miscellanea Oeconomicae* 2009, No. 13(2), p. 12.

The game theory defines the following notions: a decision maker, being the person making decisions, and a game against nature, describing a class of decision making problems where two players occur, namely the decision maker and the nature. The latter is not interested in the game outcome, therefore the decisions made by the nature are referred to as states. An optimum strategy can be chosen by applying various decision making criteria. Rules are based on different assumptions and lead to different outcomes. The decision maker is the one to set the decision making rules, determine their corresponding parameters and then, based on the outcomes previously calculated, choose the optimum action strategy[2]. Authors of various publications[3] mention a number of alternative decision making rules:

– the Wald maximin criterion – it is a conservative criterion assuming that the situation to occur will be the least advantageous one for the decision maker,

– optimistic (maximax) criterion – a criterion opposite to the Wald criterion, assuming the most optimistic state of nature to occur,

– the Savage criterion (the missed opportunities matrix) – based on a postulate of minimisation of the losses expected, resulting from the decision maker's choice, against the best possible decision for the given state of nature,

– the Laplace (Bernoulli-Laplace) criterion – choice of the strategy with the highest value of the expected outcomes resulting from each decision,

– the Hurwicz criterion – criterion choosing the best scaled value between the maximum and the minimum payoff,

– the Bayesian criterion – similar to the Laplace criterion, but assuming that probabilities in individual states of nature are known, and the average to be calculated is weighted,

– the Hodges-Lehmann criterion – a combination of the Wald and the Bayesian criteria, applying the coefficient of confidence to indicate which of the criteria is predominant[4],

– the Hurwicz-Savage criterion – a combination of the Hurwicz and the Savage criteria, where the coefficient of risk scales the value between the maximum and the minimum 'regret' from the lost opportunity matrix,

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[2] A. Bujak, Z. Śliwa, "Wybrane elementy modelu decyzyjnego (część I)", *Zeszyty Naukowe Wyższej Szkoły Oficerskiej Wojsk Lądowych* [Journal of Science of the Gen. Tadeusz Kościuszko Military Academy of Land Forces] 2003, No. 1(127), pp. 19-24; S. Krawczyk, *Zarządzanie procesami logistycznymi*, PWE, Warszawa 2001, p. 158.

[3] E.K. Zavadskas, L. Ustinovichius, F. Peldschus, "Development of software for multiple criteria evaluation", Institute of Mathematics and Informatics, Vilnius, *Informatica* 2003, Vol. 14, No. 2, pp. 265-266; Z. Jędrzejczak, K. Kukuła, J. Skrzypek, A. Walkosz, *Badania operacyjne w przykładach i zadaniach*, PWN, Warszawa 2004, pp. 238-241; E. Rotarescu, "Mathematical modeling in decision making process under conditions of uncertainty in human resources training and development", *Revista Notas de Matemática* 2011, Vol. 7(1), No. 303, pp. 49-54.

[4] E.K. Zavadskas, L. Ustinovichius, F. Peldschus, "Development of software...", op. cit., pp. 265-266.

– other, more advanced criteria, such as the proportionally weighted Laplace criterion, the proportionally weighted Laplace criterion with regrets or the nostalgic criterion[5].

In situations when the criteria are not decisive about the optimum decision, one may apply root mean square deviation as an additional criterion[6]. In practice, before the decision making criteria are applied, one often resorts to normalisation, i.e. a transformation of a table (matrix) of payoffs to values from a specific interval (e.g. [0,1]). Such transformations include the Van Delft and Nijkamp vector normalisation, Weitendorf's linear normalisation, Jüttler-Körth's normalisation, the non-linear normalisation by Peldschus et al. or the logarithmic normalisation by Edmundas Zavadskas and Zenonas Turskis[7]. Normalisation is worth being applied when the specificity of the decision making software features certain number range limitations or when one intends to transform different units into dimensionless characteristics. It should be noted that transformations (non-linear in particular) may affect the result of a solution based on different decision making criteria.

The Hurwicz criterion – one that this study is exploring more extensively – is based on application of the coefficients of optimism and pessimism. The α coefficient of optimism determines the level of the decision maker's hope to obtain the best possible outcome. The coefficients of optimism and pessimism are mutually complementary and sum up to a total of 100%. The more optimistic the decision maker is, the smaller the coefficient of pessimism (designated as $1 - \alpha$) is. The decision maker should determine the α coefficient of optimism based on individual premises, and the author of this article has made an attempt to develop a method supporting the choice of the said parameter.

A decision is made based on the selected coefficient of optimism in the following manner: the highest payoff (usefulness) is multiplied by the coefficient of optimism, whereas the lowest one – by the coefficient of pessimism. Then one is to make a decision based on a sum of the two products, choosing the maximum value and the decision corresponding to the result obtained.

The author of this article has proposed a new criterion based on Hurwicz's algorithm, one that enables making a decision when the coefficient of optimism is unknown. For a frequent issue arising from the Hurwicz criterion application is the choice of the decision maker's optimism coefficient. The method proposed may be used as an algorithm supporting the choice of the variable in question.

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[5] C. Ioan, C.A. Ioan, A. Ioan, "New methods in mathematical management of organization", *Acta Universitatis Danubius*: *Oeconomica* 2008, No. 1, pp. 34-43.

[6] C.A. Ioan, G. Ioan, "A method of choice of the best alternative in the multiple solutions case in the Games Theory", *Journal of Accounting and Management JAM* 2011, Vol. 1, No. 1, p. 3.

[7] E.K. Zavadskas, Z. Turskis, "A new logarithmic normalization method in games theory", Institute of Mathematics and Informatics, Vilnius, *Informatica* 2008, Vol. 19, No. 2, pp. 305-306.

2. Introduction to the Hurwicz criterion

The very idea behind the Hurwicz criterion application has already been explained in the introductory section, whereas the relevant strategy has been described in a formalised manner below. The following designations have been used in this article:

– dec_i – the i^{th} decision $i \in [1, \ldots, n]$ (where n – number of decisions),

– $state_j$ – the j^{th} state of nature, with the reservation that $j \in [1, \ldots, m]$ (where m – number of states of nature),

– $payoff_{ij}$ – payoff for the i^{th} decision on the j^{th} state of nature, A matrix composed of the $payoff_{ij}$ elements is referred to by numerous authors as a payoff matrix (see Table 1),

– max_i – maximum value of the i^{th} decision (formally: $max_i = Max\ [payoff_{i1}, payoff_{i2}, \ldots, payoff_{im}]$),

– min_i – minimum value of the i^{th} decision ($min_i = Min\ [payoff_{i1}, payoff_{i2}, \ldots, payoff_{im}]$).

Table 1. General payoff matrix for decision making problems

	$State_1$	$State_2$	…	$State_m$
dec_1	$payoff_{11}$	$payoff_{12}$	…	$payoff_{1m}$
dec_2	$payoff_{21}$	$payoff_{12}$	…	$payoff_{2m}$
⋮	⋮	⋮	⋱	⋮
dec_n	$payoff_{n1}$	$payoff_{n2}$	…	$payoff_{nm}$

Source: authors' own study.

The Hurwicz criterion is about determining the d_i quantity corresponding to each ($i \in [1, \ldots, n]$), dec_i decision applying the following formula:

$$d_i = \alpha\ max_i + (1 - \alpha)\ min_i \qquad (1)$$

where $\alpha \in [0, 1]$ is the coefficient of optimism chosen by the decision maker. In the understanding of the Hurwicz criterion, an optimum decision will be determined by the maximum value of quantity d_i.

One should note that, in some cases, it may happen that a certain decision is dominated by another[8]. Dominated decisions are ones whose payoff values are smaller than those of the dominating decision. And regardless of the criterion choice, it causes the dominated decision never to be chosen, hence for the sake of

[8] *Badania operacyjne*, ed. E. Ignasik, PWE, Warszawa 2001, p. 237.

simplicity of calculations, the dominated decisions are rejected even before the decision making process begins.

Analogically, when a decision is being made based on the Hurwicz criterion exclusively, one should also indicate the decisions that will never be made. Since the Hurwicz algorithm only relies on maximum and minimum values, the condition to reject the dominated decisions will be less restrictive than in the general case (since it is enough to make a reference to the maximum and the minimum value). Such a decision assumes the following form:

Definition 1 (Hurwicz's dominated decision). Let us assume there are two decisions, i.e. dec_x and dec_y. Decision dec_x will be referred to as dominated by dec_y according to Hurwicz only and exclusively when:

I: $max_y \geq max_x$ and $min_y \geq min_x$ and

II: ($max_x \neq max_y$) or ($min_x \neq min_y$),

where: max_x and min_x are the maximum and the minimum values of decision dec_x, whereas max_y and min_y are the maximum and the minimum values of decision dec_y.

Condition I means that, for any coefficient of optimism α, the dominated decision must never be made. Figure 1 illustrates an example of the criterion value given by formula (1) for decision dec_y is always greater than that of decision dec_x, whereat the Hurwicz criterion will always indicate dec_y. Consequently, regardless of the choice of the α coefficient of optimism, the decision marked as dec_x is dominated and will never be made.

It should be noted that condition II is not equivalent with $dec_x \neq dec_y$, where two decisions may assume identical maximum and minimum values, whereas they will differ in others. Condition II has been introduced in order to reject

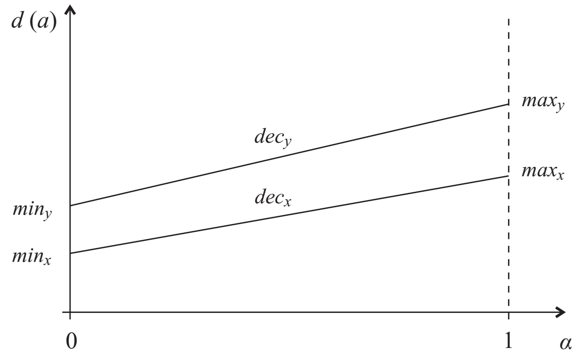

Figure 1. Example of the dominating (dec_y) and the dominated decision (dec_x), diagram of dependence between criterion function d according to formula (1) and coefficient α

Source: authors' own study.

a situation when two decisions have identical minimum and maximum values. From the perspective of the Hurwicz criterion, both decisions are equally advantageous in such a case, and so it is all the more difficult to determine if one of them pays better than the other.

With regard to the aforementioned condition (two decisions having identical maximum and minimum values), a sufficient criterion verifying which of these decisions is better is the Laplace criterion, for it indicates which decision features a higher total value of payoffs from other states of nature, meaning that the values between the maximum and the minimum one are also to be taken into account.

3. The Hurwicz criterion for different coefficients of optimism α

In practice, when decision makers apply the Hurwicz criterion, they should express their inclination to risk assuming the form of the α coefficient of optimism prior to the calculations. However, standard manuals of operational studies elaborating upon the games against nature[9] provide no guidelines on how to choose the coefficient value. In the literature of the subject[10], an example of the optimism coefficient choice is assuming the range from 0 to 1 between the minimum and the maximum value of the outcome possible to obtain, and then setting an acceptable payoff and converting it onto the scale envisaged. However, even when the decision maker can explicitly state their aversion or predilection for risk in a numerical form, it does not ensure that the choice to be made will be the most optimum one.

In order to examine the influence of the coefficient of optimism on the outcomes of application of the Hurwicz criterion, the following case has been studied. The decision maker chose coefficient $\alpha = 30\%$ and it determined the choice of decision B. However, a continuous analysis was conducted for a coefficient of optimism assuming the value from 0 to 100%, and it occurred that for the case examined:

– $\alpha \in [0; 29.99\%]$, whereat the Hurwicz criterion favoured decision A,
– $\alpha \in [29.99\%, 30.01\%]$, whereat the Hurwicz criterion favoured decision B,
– $\alpha \in [30.01\%, 100\%]$, whereat the Hurwicz criterion favoured decision C.

Therefore, if the decision marker had chosen the coefficient of optimism of 29.98% or 30.02%, it would have determined the choice of decisions A and C

[9] Z. Jędrzejczak, K. Kukuła, J. Skrzypek, A. Walkosz, op. cit., pp. 238-241; *Badania operacyjne*, op. cit., pp. 158-161.

[10] Z. Redziak, op. cit., p. 13.

respectively. It implies that even a small variation in the coefficient of optimism exerts a significant influence on the decision to be made. Whereas for a moderately optimistic decision maker, one who would have picked the coefficient of inclination to risk on the level of 70%, it is clear that, even if one had assumed a 20% error for divergence from that value, the criterion would have still pointed at decision C. The outcome obtained due to application of the Hurwicz criterion for the value of 70% is far more explicit and valuable than that obtained for α of ca. 30%. Consequently, one should analyse the α coefficient across the entire interval of values in the case when:

– there are no premises as to the choice of the coefficient of optimism,

– the coefficient of optimism is known, but the analysis is conducted in order to legitimise the outcome or to detect a situation when a minor change of the coefficient will lead to a different decision.

It should also be stressed that, in the foregoing example, the intervals of the coefficient of optimism are closed and overlapped. It results from the fact that values of d_i – compare formula (1) – are equal at the interval boundaries (at point $\alpha = 29.99\%$, $d_A = d_B$ and for $\alpha = 30.01\%$, $d_B = d_C$ correspondingly), and hence at those points the Hurwicz criterion does not decide about the optimum decision.

Authors[11] comment upon the points at which a minor change in the coefficient of optimism causes a decision shift, being the values causing instability of the solution. Therefore, instead of analysing a fixed coefficient of optimism, it is better to review a certain interval.

For the sake of an analysis of the impact exerted by coefficient α on the decision making process, one should first analyse the d_i quantity given by formula (1). In order to indicate the dependence between d_i and coefficient α, the former is to be noted in the form of a function:

$$d_i(\alpha) = a_i\, \alpha + b_i \tag{2}$$

where:

$$\begin{aligned} a_i &= max_i - min_i \\ b_i &= min_i \end{aligned} \tag{3}$$

Linear function $d_i(\alpha)$ for the i^{th} decision connects points of coordinates $(0, min_i)$ and $(1, max_i)$ within the interval of $\alpha \in [0,1]$. The maximum value of function $d_i(\alpha)$ once all decisions i have been made, for any chosen coefficient of optimism α from the given interval, reflects the result of the Hurwicz criterion.

[11] R. Guillaume, G. Marques, C. Thierry, D. Dubois, *Seeking Stability of Supply Chain Management Decisions under Uncertain Criteria*, 9th International Conference of Modeling, Optimization and Simulation – MOSIM'12, Bordeaux 2012, p. 2.

The influence of the Hurwicz criterion based on criterion function $d_i(\alpha)$ has been illustrated in Figure 2. The criterion indicates a decision for which $d_i(\alpha)$ is higher. Therefore, from the diagram, one can read that for $\alpha \in [0, \alpha_{gr}]$ decision dec_x is optimum, whereas within the interval of $\alpha \in [\alpha_{gr}, 1]$ – it is decision dec_y.

The point which separates two intervals, marked as α_{gr}, may be found by seeking a point of intersection between two straight lines $d_i(\alpha)$, and it is given by the following formula:

$$\alpha_{gr} = (b_y - b_x)/(a_x - a_y) = (min_y - min_x)/(max_x - max_y + min_y - min_x) \tag{4}$$

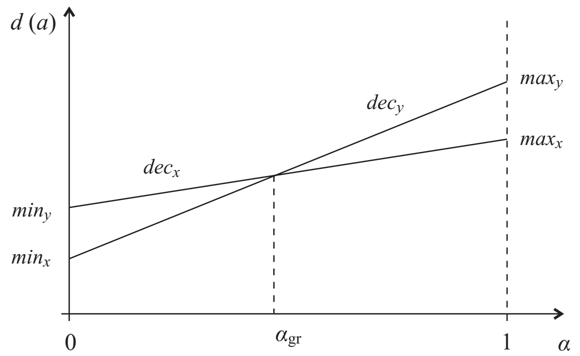

Figure 2. Diagram of dependence between criterion function $d(\alpha)$ given by formula (1) and coefficient of optimism α, with decisions dec_x and dec_y and their intersection point at α_{gr} marked

Source: authors' own study.

Moreover, the $d_i(\alpha)$ criterion function is characterised by the following properties:

Property 1: $d_i(\alpha)$ is a non-decreasing function. It is obvious that, for any chosen function $max_i \geq min_i$, therefore, in accordance with formula (3), gradient $a_i = max_i - min_i \geq 0$.

Property 2: Let us consider two undominated decisions dec_x, dec_y, and the corresponding criterion functions $d_x(\alpha)$ and $d_y(\alpha)$. If $max_x > max_y$ then linear functions $d_x(\alpha)$ and $d_y(\alpha)$ must intersect within the interval of $\alpha \in [0,1]$, and dependence $min_x \leq min_y$ is additionally satisfied.

An interpretation of property 2 is easy to illustrate in a graphical form. Let us first note that it is demanded of two decisions that they are not dominated, and hence one rejects the situation shown in Figure 1. Moreover, it is known that maximum values of both decisions differ, and therefore only two situations depicted in Figure 3 are possible.

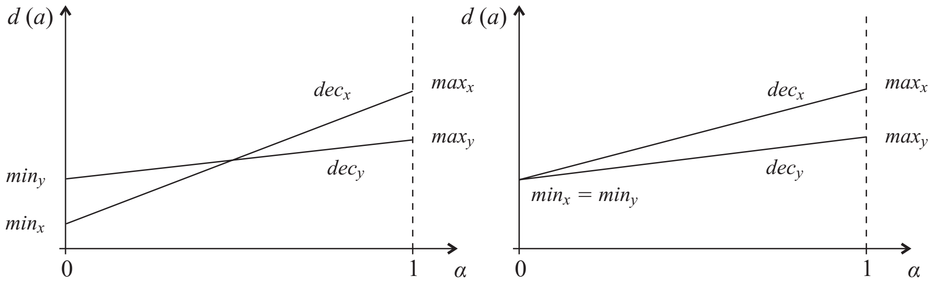

Figure 3. Diagrams of dependence between criterion function $d(\alpha)$ given by formula (1) and coefficient of optimism α, with decisions dec_x and dec_y intersecting: within the interval of $\alpha \in (0,1)$ in the left diagram, and on $\alpha = 0$ in the right diagram

Source: authors' own study.

The left diagram illustrates intersecting decisions dec_x and dec_y. Since it was assumed that $max_x > max_y$, then consequently dependence $min_x < min_y$ occurs. However, the equality condition results from an extreme position intersection of the decision at point $\alpha = 0$ (Figure 3 in the right diagram), meaning that if $max_x > max_y$ then $min_x = min_y$. By combining both dependences, one obtains property 2.

Theorem: Arranging undominated decisions – functions $d_i(\alpha)$ – according to non-decreasing coefficients max_i sets the criterion functions in an ascending order according to gradient a_i. In mathematical terms, assuming a set of decisions dec_i, and ∈ [1, …, n] and the corresponding functions $d_i(\alpha)$, if:

$$max_1 \le max_2 \le max_3 \le \ldots \le max_{n-1} \le max_n \tag{5}$$

then:

$$a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n \tag{6}$$

The proof of the above theorem is as follows: let the dec_i set of decisions be arranged analogically to formula (5):

$$max_1 \le max_2 \le \ldots \le max_y \le max_x \le \ldots \le max_{n-1} \le max_n, \tag{7}$$

then from the above arrangement, two decisions, marked with indices *x* and *y,* are selected for further considerations. Referring to dependence 2, one may assume that if $max_y < max_x$ then $min_y \ge min_x$, and if additionally $\Delta = min_y - min_x$ then $\Delta \ge 0$. From the left side of dependence $max_y < max_x$ one can subtract any chosen number greater than or equal to 0, for instance Δ. Then the condition of $max_y - \Delta < max_x$ is still satisfied, and hence $max_y - min_y < max_x - min_x$. Thus one succeeded in evidencing that if $max_y < max_x$ then $max_y - min_y < max_x - min_x$, and in order to extend the dependence to obtain a *less than* or *equal to* relation, as in formula (7), one must consider this condition when $max_y = max_x$, and such a dependence has been illustrated in Figure 4. In order to maintain the consequence analogically to Figure 3 (left diagram), the value of min_y is greater than min_x.

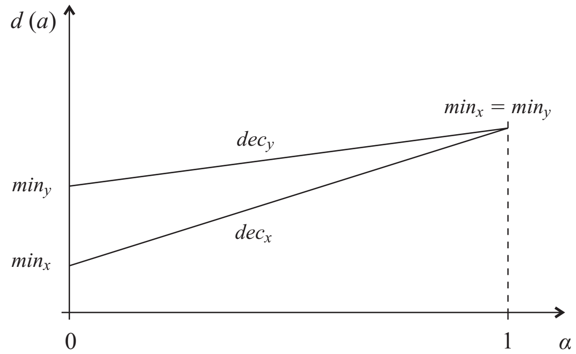

Figure 4. Diagrams of dependence between criterion function $d(\alpha)$ given by formula (1) and coefficient of optimism α, with decisions dec_x and dec_y intersecting at point $\alpha = 1$

Source: authors' own study.

According to Figure 4, equation $max_x = max_y$ implies that $min_y > min_x$, and therefore $-min_y < -min_x$, and since the maximum values are equal, then $max_y - min_y < max_x - min_x$. To recapitulate the above consideration, if $max_y \leq max_x$ then $max_y - min_y < max_x - min_x$.

An analogical reasoning may be provided for any chosen pair of inequalities from formula (7). Therefore dependence (7) can be extended to the following form:

$$max_1 - min_1 < \ldots < max_y - min_y < max_x - min_x < \ldots < max_n - min_n \tag{8}$$

Formula (3) implies that a difference between the maximum value and the minimum one determines the gradient of straight line $d_i(\alpha)$, and hence by substituting (3) to (8) one obtains:

$$a_1 < \ldots < a_y < a_x < \ldots < a_n \tag{9}$$

which concludes the proof.

One should note the fact that condition $max_x > max_y$ from property 2 guarantees difference between both decision making functions. Therefore, prior to calculations with application of the Hurwicz criterion, besides the dominated decisions, the decision maker should also eliminate those whose values are equal ($max_x = max_y$, $min_x = min_y$). Even when the decisions differ in the remaining values, the Hurwicz criterion will not decide which choice is better as it only takes the extreme values into account.

It should also be stressed that the extreme conditions from Figure 4 ($max_x = max_y$) and the left diagram in Figure 2 ($min_x = min_y$) have no practical application from the perspective of the Hurwicz criterion, and they only serve the purposes of the proof. In both cases, one of the decisions is always dominating, whereas at the intersection point, the Hurwicz criterion will not suffice to decide which of the decisions is the optimum one. Hence the decisions thus dominated can also be rejected even before the analysis starts.

A consequence of theorem 1 is the property illustrated in Figure 5. Points α_{ij} are intersections of criterion functions $d_i(\alpha)$ and $d_j(\alpha)$ corresponding to decisions dec_i, dec_j. Let us notice that, even though dec_1 features shared points with all the other criterion functions, then after the arrangement, the first intersection always indicates a decision change according to the Hurwicz criterion.

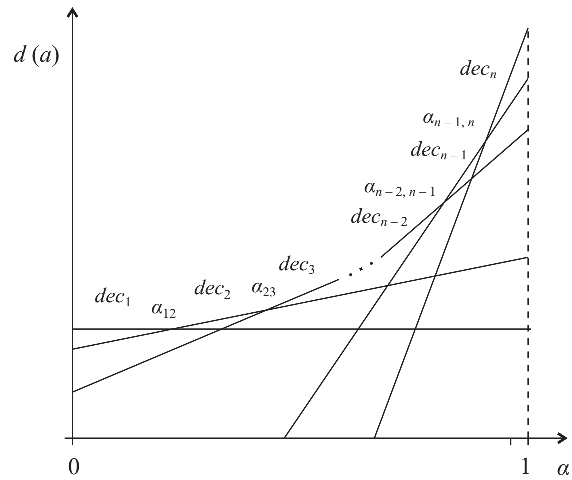

Figure 5. Diagrams of dependence between criterion function $d(\alpha)$ given by formula (1) and coefficient of optimism α, with the decisions being arranged according to ascending gradients

Source: authors' own study.

A consequence of theorem 1 is the fact that, for any chosen dec_1 indicated by the Hurwicz criterion, in order to indicate the end of the interval of the α_{ij} coefficient of optimism suggesting a choice of decision dec_1 and moving to successive decision dec_j, it is enough to find the nearest intersection (on the right-hand side) between dec_1 and the successive dec_j.

4. Algorithm of the extended Hurwicz criterion

The concept and the procedure of the extended Hurwicz criterion can be described as follows:

1. Applying the Hurwicz criterion, calculate which decision is indicated by the α coefficient of optimism from the entire interval from zero to one. Note the result in the following form:

– $\alpha \in [0, \alpha_1]$ indicating decision dec_1,
– $\alpha \in [a_1, \alpha_2]$ indicating decision dec_2,
– $\vdots$,
– $\alpha \in [\alpha_{k-1}, 1]$ indicating decision dec_k.

Select the decision with the largest interval of the coefficient of optimism.

2. The first step of the extended Hurwicz criterion is made by application of an algorithm the input of which, as in any other game against nature, is a payoff matrix (see Table 1), whereas the output is a set of intervals of the optimism coefficient values with decisions assigned. The algorithm comprises the following sequence of steps.

Algorithm for calculation of intervals:

1. Arrange the decisions according to maximum values (max_l) in a descending order, thus forming a vector of arranged decisions [dec_1, dec_2, …, dec_n].

2. Create an auxiliary table (Table 2) for dimension $n \times n$, where n is the number of decisions (the table dimension including headings is $n + 1 \times n + 1$). Intersection points of individual decisions are expressed through formula $\alpha_{ij} = (min_j - min_i)/(max_i - max_j + min_j - min_i)$.

Table 2. Auxiliary table of the extended Hurwicz criterion showing coordinates of α intersections of function $d(\alpha)$ for the corresponding decisions dec_i, $decj_j$

Decisions	dec_1	dec_2	dec_3	…	dec_{n-1}	dec_n
dec_1	×	α_{12}	α_{13}	…	$\alpha_{1, n-1}$	α_{1n}
dec_2	×	×	α_{23}	…	$\alpha_{2, n-1}$	α_{2n}
dec_3	×	×	×	…	$\alpha_{3, n-1}$	α_{3n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
dec_{n-1}	×	×	×	…	×	$\alpha_{n-1, n}$
dec_n	×	×	×	…	×	×

Source: authors' own study.

3. The analysis of Table 2 begins with the first decision ($dec_w = dec_1$), i.e. the first line ($w = 1$). The coordinate determining the preceding intersection is to be set to zero ($\alpha_{pop} = 0$). Reset the set of solutions.

4. In Table 2, find the minimum in line w (dec_w), and mark the minimum value as α_{wj}. If the minimum value repeats itself, then choose the rightmost column (j – column number, w – line number) for α_{wj}. Add the remaining minimum values (j' – column number) to the set of solutions: $\alpha = \alpha_{wj'}$ indicates decision $dec_{j'}$. Then add the following interval to the set of solutions: $\alpha \in [\alpha_{pop}, \alpha_{wj}]$ indicating decision dec_w.

5. If it is not the case that the end $j < n$ (there are still some lines), move to line j ($w = j$), and assign $\alpha_{pop} = \alpha_{wj}$ to the previous intersection. Whereas if the end $j = n$ (moving to the last line), then assign the following to the set of solutions: $\alpha \in [\alpha_{wj}, 1]$ indicating decision dec_n.

As a result of the algorithm application, one obtains a sequence of intervals of the coefficient of optimism with indications of corresponding decisions. The correctness of the foregoing algorithm is a direct consequence of theorem 1. Arrangement according to maximum values guarantees that, for the given decision making straight line (dec_w), $d_w(\alpha)$ the smallest α indicates passing on another decision (compare Figure 5), which explains the concept of the algorithm application (step 4).

The first step is the arrangement of decisions in an ascending order of maximum values, this being a prerequisite to apply theorem 1. In step two, points of intersection of individual straight lines are collated. Table 2 is symmetrical towards the elements of the main diagonal, therefore it is enough to add the missing elements above the main diagonal. The symmetry results from the fact that, regardless of the choice of the sequence of straight lines, they will still intersect at the same point. The third step is determination of the initial conditions: starting with the first decision and the first interval beginning at 0. The solution must always feature decision dec_1 (one with the smallest maximum) and decision dec_n (one with the largest maximum) which determines the emergence of intervals $\alpha \in [0, \alpha_i]$ and $\alpha \in [\alpha_i, 1]$ (step 3 and 5). Step four indicates that the subsequent decision is the one with the smallest point of intersection with the one currently being examined (generator 1). Step four entails a situation when the same minimum value occurs several times, and this condition has been illustrated in Figure 6. Point $\alpha_{ij} = \alpha_{ik} = \alpha_{il}$ is the one at which the Hurwicz criterion does not settle which of the decisions is the best. Therefore, one should either state that there is no solution, or indicate that, for the given α_{ij} coefficient of optimism, the criterion indicates several decisions at the same time. As shown in Figure 6, when: $\alpha \in [0, \alpha_{ij}]$ then dec_i, $\alpha = \alpha_{kj}$ then dec_k, $\alpha = \alpha_{il}$ then dec_l, $\alpha \in [\alpha_{ij}, 1]$ then dec_j. So it has been noted in step 4.

The stop condition (step 5) arises directly from an analysis of Table 2, and moving to line n implies the end, since the given decision no longer intersects with any other on the right-hand side.

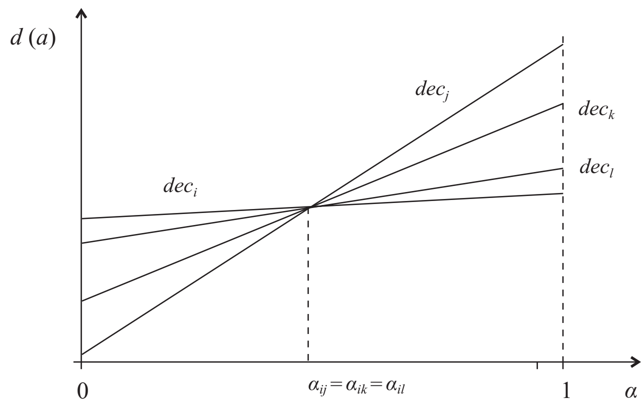

Figure 6. Diagrams of dependence between criterion function $d(\alpha)$ given by formula (1) and coefficient of optimism α, with decisions dec_i, dec_j, dec_k, dec_l intersecting at the same point

Source: authors' own study.

An advantage of the foregoing algorithm based on theorem 1 is the simplicity of implementation in computer systems. A similar solution is possible to obtain by a graphical method, by comparing individual decision making functions $d_i(\alpha)$ with function $f(\alpha) = max\ [d_1(\alpha), d_2(\alpha), \ldots, d_n(\alpha)]$ on $\alpha \in [0, 1]$. Such a solution will be satisfactory depending on the preset level of error margin. In formal terms, in order to inform the computer which coefficients of optimism indicate which decisions with index i, one would need to solve $\alpha \in f(\alpha) \cup d_i(\alpha)$ or conduct a complicated analysis of intersections between individual criterion straight lines. To implement the foregoing solution in a low-level programming language would be very complicated, and therefore it is better to apply the algorithm of interval calculation based on theorem 1.

5. Application of the extended Hurwicz criterion in logistics

Decision making issues typically occur in such areas of logistics as logistics management, transport and warehousing. Among them, there are numerous spheres in which one is bound to make decisions, including normalisation, customer service, stock management, company supplies, distribution, logistic education, selection of mode and means of transport, transport security, traffic monitoring, recipient service as well as the choosing of the number, costs, cubage, ownership form, location, organisation and equipment of warehouses, and many others[12].

Let us consider a simple case of choosing the means of transport. Let us further assume that the decision maker in a small logistics company is to sign a contract with a supplier, but the quantity of materials to be shipped in the period of time envisaged is unknown. The decision maker has only access to estimated profits (expressed in PLN thousands) depending on the supplier choice and the states of nature, i.e. shipment volumes. The applicable data have been collated in Table 3.

Table 3. Payoff matrix for the sample choice of means of transport

Suppliers	Order volume			
	more than 10,000 tonnes	more than 8,000 tonnes	more than 6,000 tonnes	more than 4,000 tonnes
A	60	30	35	10
B	50	45	45	20
C	40	40	40	40

Source: authors' own study.

The Laplace criterion, which constitutes an arithmetic average in practice, does not settle whether to choose decision B or C (the average equals 40). Neither does application of the Savage criterion provide a solution, therefore the decision maker has decided to settle the outcome with the Hurwicz criterion (with the coefficient of optimism unknown).

Firstly, the interval calculation algorithm is to be applied. In the first step, one is to arrange the decisions in an ascending order according to the maximum value, i.e. $max_A = 60$, $max_B = 50$, $max_C = 40$, which means that the specific order is provided. Then one is to create an auxiliary table containing intersection points (see Table 4) – intersection values according to formula (4).

Table 4. Auxiliary table of the extended Hurwicz criterion for the case analysed

Decisions	dec_A	dec_B	dec_C
dec_A	×	0.5	0.6
dec_B	×	×	0.6(6)
dec_C	×	×	×

Source: authors' own study.

[12] E. Chylak, "Logistyka małych i średnich przedsiębiorstw", *Zeszyty Naukowe Wyższej Szkoły Cła w Warszawie* [Scientific Journals Higher School of Customs in Warsaw] 2004, No. 6, p. 3.

In the first line, the minimum value is 0.5, therefore one must add to the solution the interval of $\alpha \in [0, 0.5]$ which indicates decision dec_A. Since 0.5 is to be found in column 2, one must move to line 2. In the second line, the minimum value is 0.6(6), therefore the next interval of the solution is $\alpha \in [0.5, 0.6(6)]$ which indicates decision dec_B. Since 0.6(6) is in column 3, one must move to line 3, and since there are no more intersections, the procedure is completed by adding $\alpha \in [0.6(6), 1]$ which indicates decision dec_C. To recapitulate the foregoing:

- $\alpha \in [0, 0.5]$ indicates decision dec_A,
- $\alpha \in [0.5, 0.6(6)]$ indicates decision dec_B,
- $\alpha \in [0.6(6), 1]$ indicates decision dec_C.

The broadest interval of the coefficient of optimism indicates decision A. Therefore, the strategy conforming with the extended Hurwicz criterion is decision A. A comprehensive collation of successive steps in the interval calculation procedure along with a detailed description of variables has been provided in Table 5.

Table 5. Successive steps in the interval calculation procedure for the case examined

Step No.	Formal calculations
Step 1	$max_A = 60 > max_B = 50 > max_C = 40$, hence the arrangement of decisions dec_A, dec_A, dec_A
Step 2	$\alpha_{AB} = (min_B - min_A) / (max_A - max_B + min_B - min_A) = (20 - 10)/(60 - 50 + 20 - 10) = = 0.5$ $\alpha_{AC} = (min_C - min_A) / (max_A - max_C + min_C - min_A) = (40 - 10)/(60 - 40 + 40 - 10) = = 0.6$ $\alpha_{BC} = (min_C - min_B) / (max_B - max_C + min_C - min_B) = (40 - 20)/(50 - 40 + 40 - 20) = = 0.6(6)$
Step 3	$w = 1$, $\alpha_{pop} = 0$
Step 4	The minimum in line $w = 1$ is 0.5, hence column j = 2, $\alpha_{wj} = \alpha_{12} = 0.5$, and hence the solution: $\alpha \in [\alpha_{pop}, \alpha_{wj}] = [0, 0.5]$ indicates decision $dec_w = dec_1$
Step 5	Checking the stop condition $j = 2 < n = 3$, hence step 4, setting variables: $\alpha_{pop} = \alpha_{12} = 0.5$, $w = 2$
Step 4	The minimum in line w = 2 is 0.6(6), hence column j = 3, $\alpha_{wj} = \alpha_{23} = 0.6(6)$, and hence the solution: $\alpha \in [\alpha_{pop}, \alpha_{wj}] = [0.5, 0.6(6)]$ indicates decision $dec_w = dec_2$
Step 5	Checking the stop condition $j = 3 \in n = 3$, hence the end, adding the solution $\alpha \in [\alpha_{wj}, 1] = [0.6(6), 1]$ indicates decision $dec_n = dec_3$

Source: authors' own study.

6. Conclusions

In this article, the author has discussed the extended Hurwicz criterion based on its standard variant widely known in the game theory. The choice of a decision assumes a parametric form compared to the original version. The decision maker does not need to know the coefficient of optimism, and the criterion indicates intervals of the parameter as well as the corresponding results. The largest of the intervals implies the strategy the decision maker should assume. The very foundation of the method proposed is the interval calculation algorithm based on the properties discussed in the article. Theorem 1 shows that putting decisions in a sequence according to ascending maximum values of each successive decision results in an arrangement with the nearest intersection of the decision making straight line causing a shift to another decision (compare Figure 5). Having used that observation, one can considerably simplify the calculation of intervals indicating individual decisions.

A more extensive perception of the Hurwicz criterion, as discussed in this paper, also lead to more profound understanding of the impact exerted by the coefficient of optimism and may be used as an auxiliary tool while the decision maker is to choose the level of hope to obtain payoff. When the coefficient of optimism is unknown, one may:

– apply the extended Hurwicz criterion in the form proposed in the article, i.e. choose a strategy featuring the largest interval of the coefficient of optimism,

– apply the extended Hurwicz criterion (interval calculation algorithm) as supportive for the choice of the coefficient of optimism. Even if the coefficient of optimism is unknown to the decision maker, one can certainly find specific premises decisive of its value (pessimism or optimism). Being optimistic, one may check within the interval from 50% to 100% (a pessimist would check the interval of 0-50%) which decisions are indicated by the Hurwicz criterion, and based on such information, choose an optimum coefficient. Consequently, in formal terms, one will actually apply the standard Hurwicz criterion.

If the coefficient of optimism is known, the extended Hurwicz criterion (interval calculation algorithm) can be used by the decision maker as a tool to detect instability of the solution, namely situations when a minor change in the coefficient of optimism affects the decision choice based on the standard Hurwicz criterion. The following three main scenarios are possible:

– stable solution – far from points separating the choice of other decisions; a very good solution, as the decision maker is certain that, within a specific interval, the strategy is insensitive to the choice of the coefficient of optimism,

– solution unstable near the boundary point – the chosen coefficient of optimism lies in a close vicinity of a point which indicates a different decision. In such a case, one should increase the coefficient of optimism if the decision maker is indeed optimistic, or decrease it (for a pessimist) in order to obtain a stable solution,

– solution unstable near several boundary points – small changes in the chosen coefficient of optimism trigger a choice of a number of other decisions. If the decision maker is convinced about the chosen α, the solution should be intentionally accepted bearing the instability in mind. Otherwise, it is recommended to use different criteria, since the Hurwicz criterion leads to the result being very sensitive to the parameter choice.

Moreover, the decision maker should define the notions of **near** and **far**, or in other words, determine what (percentage) interval of the coefficient of optimism indicating the given decision is to be considered satisfactory.

The methods proposed in the article are very practical in a certain dimension, and they may be applied by an organisation to support decision making systems. Furthermore, the extended Hurwicz criterion and the interval calculation algorithm have been presented as a procedure easy to implement in computer systems.

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Rozszerzone kryterium Hurwicza podejmowania decyzji dla zastosowań logistycznych

Streszczenie. W artykule zaprezentowano nowe kryterium decyzyjne bazujące na popularnym w teorii gier z naturą kryterium Hurwicza. Zostały przedstawione własności funkcji decyzyjnej tego kryterium, zastosowane następnie do nowego algorytmu podejmowania decyzji. Wspomniane podejście może służyć również jako nowa taktyka wyboru współczynnika optymizmu lub sprawdzenia poprawności już wybranego.

Słowa kluczowe: logistyka, problemy decyzyjne, gry z naturą, kryterium Hurwicza, współczynnik optymizmu